Change Point Detection in Black-scholes Models Using Recursive Estimation

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***Abstract*-This paper is concerned with change point detection using recursive estimation in non-linear time series. Specially, we consider the Black and Scholes (1973) option pricing model under stochastic volatility assumption. The test statistic is given and we describe how its null distribution may be simulated.**

Keywords-Black-Scholes Models; Brownian bridge; Change point; Cusum; RML; SDE; Simulation

I. INTRODUCTION.

A key assumption for making inference using a statistical model is the model stability over time. This condition, however, may be violated in practice and all statistical outputs become wrong. For example, in financial time series setting, the parameters of a GARCH series may be shifted, because of changes in economic environment and forecast results fail. In this case, a structural break happened and a common technique to assess the constancy of a model is change point analysis. Change point detection in financial time series is very important, in practice. For example, the portfolios volatility may increase as risk premium rises and a change point analysis is needed. Kim et al. (2000) considered the problem of multiple change points in GARCH models. Hillebrand and Schnabl (2003) studied change point detection in volatility of Japanese foreign exchange rates under GARCH modeling. Lee et al. (2006) detected change points in a diffusion process using one step estimators. Iacus (2008) considered the change point problem in volatility of a diffusion process. An excellent reference in change point problems is Csorgo and Horvath (1997).

Both recursive estimation and recursive errors topics play important roles in statistical inference, especially in change point analysis. Plackett (1990) and Bartlett (1991) studied the recursive least square estimations in a linear regression model. Hannan (1980) considered recursive estimators based on ARMA models, for more details see Hannan (1980) and references therein. The logic behind the recursive estimation is to compute parameter estimates with a fraction of sample. If the actual parameters are fixed over the time, then these estimates are very close together. If a change (or some) has happened, then the recursive estimates will show this shift. Brown et al. (1975) used an alternative method. They derived estimated recursive residuals in a regression model and applied the cusum method. Their approach is used by Kramer et al. (1988) in dynamic models and it is continued up to this time. Throughout the current paper, w_t and B_t are two standard Brownian motions defined on $t \in [0,T]$ such that $cor(dw_t, dB_t) = \rho$

Hannan (1980) presented a regression model representation of ARMA process and used Plackett algorithm to derive the recursive estimates of regression coefficient. As follows, we review this method with two new models.

Example 1. Let X_t be a ARCH(q) model, i.e., $X_t = h_t \varepsilon_t$ where ε_t's are independent and identically distributed variables with $E(\varepsilon_1) = 0$ and $E(\varepsilon_1^2) = 1$ and

$$h_t^2 = \beta_0 + \sum_{j=1}^{q} \beta_j X_{t-j}^2.$$

Let $\eta_t = \varepsilon_t^2 - 1$. Then following Chatterjee and Bose (2005), we have $y_t = z_t{}' \beta + h_t^2 \eta_t$, where $\beta = (\beta_0, \beta_1, ..., \beta_q)'$ and $z_t{}' = (1, X_{t-1}^2, ..., X_{t-q}^2)$ and $y_t = X_t^2$. Therefore, following Brown et al. (1975), the weighted recursive estimates of β is

$$b_r = b_{r-1} + A_r^{-1} h_r^{-2} z_t{}'(y_r - z_t{}' b_{r-1}), b_0 = 0,$$

And A_r is the matrix whose i-th row is z_i satisfying in the following relation

$$A_r^{-1} = A_{r-1}^{-1} + \frac{h_r^{-2} A_{r-1}^{-1} z_r z_r{}' A_{r-1}^{-1}}{1 + h_r^{-2} z_r{}' A_{r-1}^{-1} z_r}.$$

Also, the cusum test procedure of recursive estimated residuals e_r may be applied to detect a change point, where

$$e_r = \frac{h_r^{-1}(y_r - z_r{}' b_{r-1})}{\sqrt{1 + h_r^{-2} z_r{}' A_{r-1}^{-1} z_r}}.$$

Recursive estimation also appears in maximum likelihood estimation. However, in the next example, we study it for diffusion processes.

Example 2. Let X_t be a continuous time AR(1) model given by

$$dX_t = \alpha X_t dt + \sigma dX_t.$$

Using the Euler method for a small bandwidth h, we have $Y_{t_i} = \alpha X_{t_i} + \sigma^* Z_{t_i}$, where

$$Y_{t_i} = h^{-1}(X_{t_i+h} - X_{t_i}), \sigma^* = h^{-1/2} \sigma$$

And Z_{t_i} are independent and identically distributed come form $N(0,1)$ distribution. Again, the Brown's method may be applied here to detect change points. In the next section, we see how the recursive estimation may be applied for Black-Scholes formulae.

II. BLACK-SCHOLES MODEL

Recursive estimation also appears in continuous time series. For example, Lee et al. (2006) proved that the limiting distribution of the one-step ML estimator converges to Brownian motion as sample size goes to infinity. Kutoyants (2004) derived an recursive relation for ML estimator in diffusion processes. This estimation procedure plays a central role in many fields of applications such as economic, engineering and finance (see Ljung *et al*. (1988) and Young (2011)). Test statistics in change point field (say cusum statistic) are constructed based on the RML (see Lee et al. (2006)). One of useful financial models is Black and Scholes's (1973) option pricing formula. This model has had a huge influence on growth of financial mathematics, say, for making decision in hedging or valuing European call and put options. This formula implies that the percentage change in price of a stock s_t during life time dt (too small) follows a normal distribution. More precisely, it states that

$$ds_t = \varphi s_t dt + \sigma_t s_t dw_t.$$

Here, we assume that φ is an unknown parameter which should be estimated. There are many estimation methods for φ such as the recursive maximum likelihood (RML) method. In fact, it is the maximum likelihood estimation $\hat{\varphi}_t$ based on observations $x_s, s \in [0,t]$. The standard Black-Scholes model assumes that the underlying volatility is constant over the life of stock. However, this assumption isn't true, in practice. The stochastic volatility models are solutions to this difficulty. These models consider the underlying stock's volatility as a random process, i.e.,

$$d\sigma_t^2 = \mu\sigma_t^2 dt + \zeta\sigma_t^2 dB_t.$$

In this case, it is easy to see that

$$\hat{\varphi}_t - \varphi = C_T^{-1}\int_0^t \sigma_s^{-1} dw_s,$$

Where $C_T = \int_0^T \sigma_u^{-2} du$. Therefore, the conditional distribution of $\hat{\varphi}_t - \varphi$ given C_T is normal distribution $N(0, C_T^{-1})$.

In this paper, we test the existence of a change point in parameter φ based on RML estimator $\hat{\varphi}_t$. This means parameter φ is changed at some unknown time points t_0 from φ_1 (for $t \le t_0$) to φ_2 (for $t > t_0$). To this end, following Kutoyants (2004), the maximum of likelihood function is the maximum of following process *with respect to* argument t

$$\int_0^t \frac{(ds_u - \widehat{\varphi}_{1t} s_u du)^2}{\sigma_u^2 s_u^2} + \int_t^T \frac{(ds_u - \widehat{\varphi}_{2t} s_u du)^2}{\sigma_u^2 s_u^2}.$$

Here $\widehat{\varphi}_{1t}$ and $\widehat{\varphi}_{2t}$ based on observations $\{x_s\}_{s\in[0,t]}$ and $\{x_s\}_{s\in[t,T]}$, respectively. The null hypothesis of no change point is rejected when the above test statistic is large. Following Bai (1994), we can show that the test statistic is reduced to

$$U = \min_{0<t<T} U_T(t)$$
$$= \min_{0<t<T} C_T a_t (1-a_t) \left|(\widehat{\varphi}_{1t} - \widehat{\varphi}_{2t})^2\right.,$$

Where $a_t = C_T^{-1}\int_0^t \sigma_u^{-2} du$. It can be shown that, under the null hypothesis, the test statistic is minimum of absolute value of the following process

$$\frac{1}{\sqrt{C_T a_t (1-a_t)}}[X_t - a_t X_T],$$

With $X_t = \int_0^t \sigma_u^{-1} dW_u$. The $X_t - a_t X_T$ is the diffusion type of X_t . Therefore, we can simulate the null distribution of test statistic by simulation of Bessel-type tied-down process of $X_t = \int_0^t \sigma_u^{-1} dW_u$.

Remark 1. Under the alternative hypothesis, there is a change point and $\varphi_1 \ne \varphi_2$. It is easy to see that

$\left|(\widehat{\varphi}_{1t} - \widehat{\varphi}_{2t}) - E(\widehat{\varphi}_{1t} - \widehat{\varphi}_{2t})\right|$ equals to

$$\left|\frac{\int_0^t \sigma_s^{-1} dw_s}{\int_0^t \sigma_s^{-2} dw_s} - \frac{\int_t^T \sigma_s^{-1} dw_s}{\int_t^T \sigma_s^{-2} dw_s}\right|.$$

This shows that $(\widehat{\varphi}_{1t} - \widehat{\varphi}_{2t})$ is very close to its expectation. Therefore, it behaves similar to $E(\widehat{\varphi}_{1t} - \widehat{\varphi}_{2t})$. Let the magnitude of change be $\delta = \varphi_1 - \varphi_2$. Without loss of generality, suppose that δ is positive. One can check that the conditional expectation of $E(\widehat{\varphi}_{1t} - \widehat{\varphi}_{2t})$ given $\sigma_{[0,T]} = (\sigma_t, 0 \le t \le T)$ is $\frac{1-a_{t_0}}{1-a_t}$ for $t \le t_0$ and is $\frac{a_{t_0}}{a_t}$ for $t > t_0$. Therefore,

$$\frac{1}{\delta}E(U_T(t)) = \begin{cases} E(\sqrt{C_T}(1-a_{t_0})\sqrt{\frac{a_t}{1-a_t}}) & t \le t_0 \\ E(\sqrt{C_T}\, a_{t_0}\sqrt{\frac{a_t}{1-a_t}}) & t \le t_0. \end{cases}$$

The function at the right side of above equation is increasing before t_0 and is decreasing after t_0 This shows that the minimizing of $U_T(t)$ is a consistent estimator for change point t_0 .

III. CONCLUSION

In this paper, we propose the use of recursive estimators for change point detection. Our approach is constructed by representing the non-linear models as regression type model. We consider this estimator for GARCH and diffusion series. As an important special case, the Black-Scholes model is considered and the null distribution of test statistic is simulated. This paper can be considered as generalization of

recursive analysis in discrete time-series to continuous time stochastic processes.

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